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ORIGINAL ARTICLE

The impact of potential-based physics models on pricing in energy networks

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Abstract
Pricing of access to energy networks is an important issue in liberalized energy sectors because of the natural monopoly character of the underlying transport infrastructures. We introduce a general pricing framework for potential-based energy flows in arbitrarily structured transport networks. In different specifications of our general pricing model we discuss first- and second-best pricing results and compare different pricing outcomes of potential-free and potential-based energy flow models. Our results show that considering nonlinear laws of physics leads to significantly different pricing results on networks and that these differences can only be seen in sufficiently complex, e.g., cyclic, networks as they can be found in real-world situations.

Keywords Energy networks · Pricing · Gas networks · Electricity networks

JEL Classification C61 · L94 · L95

1 Introduction

Most industrialized countries have liberalized their energy sectors in the last decades. As opposed to trading, which takes place at competitive markets, energy transport networks still have the character of natural monopolies. This is the reason why certain regulation rules have to be imposed for network access. Thus, an important topic is

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the pricing of network access, which should reflect the costs of transportation and additional costs arising from investment in capacity expansion of the network.

The economic situation is similar for different types of energy like electricity or natural gas. In most countries with liberalized energy sectors, supply and demand is traded on markets and the resulting quantities have to be transported through the corresponding networks. The laws of physics that govern the energy flow through these networks differ but have some mathematical commonalities. However, many publications on spatial energy economics abstract from these, often nonlinear, physics models because they focus on other aspects like market power. The mentioned physics models of energy flows are mainly determined by potential gradients, where potentials are voltage angles, gas pressures, or hydraulic heads in electricity, gas, and water networks, respectively. In this paper we discuss these generic physics models that can be used to model energy flows in electricity, natural gas, and drinking water networks and thus present a unified framework to discuss pricing issues for different types of energy networks. We are aware that these sectors need different economic models on top of the physical constraints that we, in this paper, embed into a very basic economic modeling. In particular, we compare both an energy flow model that abstracts from potentials with its potential-based counterpart and analyze the impact of physics on a simple economic pricing model that is based on dual variables of the respective clearing constraints. To the best of our knowledge, this is the first contribution addressing these different network flow models in an integrated pricing framework. Thus, our contribution is to present and discuss the general effects of potential-based physics models on pricing—without considering the effects that arise due to specific economic modeling for the mentioned energy sectors like electricity or natural gas.

There are several specifications of pricing models in the literature of energy economics. In the present paper, we consider first- and second-best pricing frameworks. Here, the first- and second-best pricing model differ in the fact whether a break-even constraint for the transport operator (TO) is incorporated or not. The investigation of both models and their differences is of special importance in networks like, e.g., gas networks, where short-run costs (mainly arising from operating gas compressors) only represent a small amount of the overall costs. In such a case, the consideration of second-best pricing is necessary in order to cover short-run network operation and long-run investment costs.

There exists exhaustive literature on access pricing for energy networks, especially for electricity and gas networks. Examples for studies of access pricing for natural gas networks include, e.g., Cremer and Laffont (2002), Cremer et al. (2003), or Meran et al. (2010). They discuss pricing issues in perfectly competitive gas markets and the case of market power. Lochner (2011) studies the valuation of gas transport infrastructures and the identification of congestion in the European gas market. As the previous mentioned papers, it uses potential-free gas flow models. The presented case studies are based on networks with no more than three nodes. Additionally, studies of specific real-world gas pipelines, i.e., two-node networks, are given in the literature; see, e.g., Micola and Bunn (2007). Other examples for gas market studies using potential-free models on two- or three-node networks include, e.g., Gasmi and Oviedo (2010) and Rosendahl and Sagen (2009). For contributions with an economic focus that also consider pressure loss constraints see, e.g., Midthun et al. (2009), where an economic dispatch pricing of natural gas is considered or Midthun et al. (2015), where a capacity investment model for natural gas is discussed. Multilevel equilibrium models for gas markets with nonlinear physics models can also be found in Grimm et al. (2018), (2019a). We also refer to the latter of the last two papers for a more detailed review of the relevant literature on gas markets.

There also exist many contributions regarding electricity networks. The majority of studies are based on DC power flow models, in which node potentials are taken into account; see, e.g., Cardell et al. (1997), Chao and Peck (1996, 1998), Deng and Oren (2001), Hogan (1992, 1997), Joskow (1997), Léautier (2001) and Schweppe et al. (1988); to name only a few.

Let us also mention that access pricing to networks is not restricted to energy networks; see, e.g., Laffont and Tirole (1994) for access pricing to telecommunication and railway networks, which also exhibit a natural monopoly character.

Finally, second-best pricing is used in many studies about natural monopolies like in transportation (Arnott and Yan 2000; Verhoef 2002; Winston 1985), water (Johansson et al. 2002; Spulber and Sabbaghi 2012), or electricity networks (Joskow 2008).

In the context of this literature, our contributions are the following:

(1) We state general first- and second-best pricing frameworks on arbitrary networks.
(2) We compare pricing models that abstract from potential laws with models that incorporate (possibly) nonlinear and thus nonconvex potential-based energy flows.
(3) A specific instantiation of our models that neglects transportation costs is analyzed both for the potential-free and the potential-based model and the existence of price zones for the former case is proven using first-order optimality conditions of an alternative flow formulation.

Let us note that we are aware of the fact that we abstract from many practically important issues like, e.g., market power. We discuss the possible combination of economic market power models with potential-based physics models later in the conclusion.

Our main results are the following: First, we discuss the properties of pricing for potential-free energy flow models in transport networks. The main property is that most of the results are local pricing rules, i.e., flow situations in some region of the network do not influence the pricing in other regions. This is shown to be the case both for first- and second-best pricing. Second, we show that this does not hold for potential-based flow models in networks. Here, binding bounds on capacities or potentials have a significant impact on pricing everywhere in the network, which is also true both for first- and second-best pricing. The implications of these results are double-edged. On the one hand, it shows that cost-reflective pricing seems to be a goal that is very hard to reach if one considers potential-based flow models. On the other hand, the value of capacity investments in potential-based energy flow models depends on the specific flow situation. Thus, reasonable policy implications only seem to be possible if multi-scenario settings are considered, which typically leads to much harder optimization models to solve—in addition to the already complicating nonlinearities.

The paper is organized as follows. Section 2 collects the notation required throughout the paper. Afterward, Sect. 3 analyzes first- and second-best pricing outcomes on networks for the case of potential-free flow models and Sect. 4 then discusses the results for the potential-based counterparts. Section 5 analyzes the case without transportation costs on arcs both for the potential-free and potential-based case. In Sect. 6 we present a specific example to illustrate the effects of potential-based flows and discuss the differences with respect to the potential-free counterparts. Finally, Sect. 7 summarizes and states some topics for future research.

2 Technical and economical setup

We consider energy transport networks that we model using directed graphs $G = (V, A)$ with node set V and arc set A. For what follows, we assume that all graphs are connected. Nodes are denoted by $u, v \in V$ and correspond to points in the network where supply $s_u \geq 0$ and demand $d_u \geq 0$ is located. Supply costs are modeled by strictly convex functions $c_{\mathrm{op},u} = c_{\mathrm{op},u}(s_u)$. These costs are assumed to be nonnegative and strictly increasing, i.e., $c_{\mathrm{op},u} \geq 0$, $c'_{\mathrm{op},u} > 0$. Demand is modeled by inverse demand functions $p_u = p_u(d_u)$ that satisfy the following assumptions: Their price-intercept is positive, i.e., $p_u(0) > 0$, and they are strictly decreasing, i.e., $p'_u < 0$. Elasticity of supply and demand at node $u \in V$ is denoted by E_u^s or E_u^d, respectively.

Flows on arcs $a \in A$ are denoted by q_a. They can be positive or negative, yielding, together with the direction of the arcs $a = (u, v) \in A$, the direction of flow: Positive flow $q_a > 0$ indicates flow from u to v whereas negative flow $q_a < 0$ indicates reverse flow from v to u. Additionally, we make use of the standard δ-notation, i.e., $\delta^{\mathrm{in}}(u)$ and $\delta^{\mathrm{out}}(u)$ are the sets of in- and outgoing arcs of node u: $\delta^{\mathrm{in}}(u) := \{a \in A : a = (v, u)\}$, $\delta^{\mathrm{out}}(u) := \{a \in A : a = (u, v)\}$.

Depending on the specific model under consideration, arc flows are limited by two different types of constraints. In simple linear models, they are restricted by capacities $K_a > 0$, i.e., $-K_a \leq q_a \leq K_a$ for all $a \in A$. In more accurate models, arc flows are coupled to node potentials ℓ_u, $u \in V$, and are governed by specific laws of physics. In this case, we use the generic relation

$$\ell_v = \ell_u - W_a(K_a) f(q_a). \tag{1}$$

Here, f is a possibly nonlinear flow function and W_a is a potential loss function that depends on a specific type of capacity; see below. Throughout the paper, we make the following assumptions: W_a is positive and strictly decreasing, i.e., $W_a > 0$, $W'_a < 0$. The flow functions are strictly increasing, i.e., $f' > 0$ and have the same sign as the flow itself, i.e., $\operatorname{sgn}(f(q_a)) = \operatorname{sgn}(q_a)$.

In order to demonstrate the generality of Equation (1), we discuss the cases of gas, water, and electricity transport networks; see also Groß et al. (2019). For modeling gas flow through pipeline systems, one often uses a quadratic approximation of the relation between node potentials and flows on arcs. In this case, the potentials are squares of gas pressures and the flows are gas mass flows. The flow function is given by $f(q_a) = |q_a| q_a$ and the pressure loss term is defined as

$$W_a := \left(\frac{4}{\pi}\right)^2 \frac{\kappa_a}{{D_a}^5}, \tag{2}$$

where $\kappa_a > 0$ summarizes some physical and technical parameters like the friction at the inner pipe wall and D_a is the diameter of the pipe. The latter is the main design parameter in gas pipeline systems and thus serves as the "capacity" K_a in this case. For more information on modeling gas flow in pipelines, see Koch et al. (2015) and, in particular, the chapter by Fügenschuh et al. (2015) therein.

In water networks, potentials correspond to hydraulic heads and the potential loss is often modeled using the so-called Hazen–Williams equation, in which the flow function is given by $f_a(q_a) = \operatorname{sgn}(q_a)|q_a|^{1.852}$ and the hydraulic loss term reads $W_a = \kappa_a / {D_a}^{4.87}$, where κ_a again summarizes some technical parameters of the pipe and D_a is its diameter. For more details on hydraulic head loss modeling in water networks; see Larock et al. (2010). We are aware of the fact that fresh water is not comparable from an economic perspective to gas or electricity that are often traded on liberalized markets. However, we also give this example here to illustrate the generality of the potential-based model of flows in networks.

In order to satisfy the regularity assumption on f (e.g., the existence of the first derivative), one often uses smoothings of the original nonsmooth flow functions; see Burgschweiger et al. (2009) for water and Schmidt (2013); Schmidt et al. (2015, 2016) as well as the references given above for modeling of gas flow in networks.

Electricity networks are often modeled using the so-called DC model, in which arc flows correspond to power flows and node potentials are voltage angles; see, e.g., Chao and Peck (1996, 1998), Joskow (1997), and Schweppe et al. (1988) for a discussion in economic frameworks. Here, the flow function is simply given by $f(q_a) = q_a$ and the potential loss term is given by $W_a = 1/B_a$, where $B_a > 0$ is the susceptance of the arc; see, e.g., Kirschen and Strbac (2004). In contrast, AC power flows do not fit into our framework.

We finally note that our unified framework is not restricted to convex flow models. For instance, the cases of gas and water transport networks lead to nonlinear and nonconvex flow problems.

Arc capacity investment costs are denoted by $c_{\mathrm{inv},a} = c_{\mathrm{inv},a}(K_a)$ and assumed to be convex, nonnegative, and strictly increasing, i.e., $c_{\mathrm{inv},a} \geq 0$, $c'_{\mathrm{inv},a} > 0$. Furthermore, convex transportation costs in dependence of flow is denoted by $c_{\mathrm{op},a} = c_{\mathrm{op},a}(q_a)$. Here, we make the assumptions $c_{\mathrm{op},a} \geq 0$ and $c'_{\mathrm{op},a}(q_a) q_a \geq 0$.

Finally, we fix some technical notation: Vectors are given by the corresponding symbols without node or arc index, e.g., $d := (d_u)_{u \in V} \in \mathbb{R}^{|V|}$ denotes the vector of all demands. For better reading, primal variables are always denoted by Roman letters whereas dual variables are denoted by small Greek letters.

3 Pricing with potential-free network flow models

In this section we derive and state potential-free pricing models on general transport networks. To this end, we start describing the optimization problems of the separate players that act on the market. This leads to a generalized Nash equilibrium problem (GNEP) on a network. Such GNEPs are often used for modeling strategic interaction and usually possess infinitely many solutions. Due to the latter multiplicities we follow the approach to choose the specific solution that is also a variational equilibrium. This variational equilibrium can then be computed by a mixed complementarity problem (MCPs) if the convexity assumptions hold as stated in Sect. 2. By doing so, we also obtain a model of a perfectly competitive market. MCPs have become the standard mathematical tool for modeling equilibria in liberalized energy markets; see, e.g., the book Gabriel et al. (2012) and the many references therein. Moreover, these MCPs are known to be equivalent to a single welfare optimization problem for models of perfectly competitive markets. Thus, the variational GNEP solutions can also be analyzed by studying the corresponding welfare optimization problem. This is exactly what we do to analyze the resulting pricing mechanisms. Moreover, we extend this welfare optimization problem by a break-even constraint in order to model second-best pricing. The special case without transportation costs on arcs is discussed later in Sect. 5.1.

3.1 First-best pricing

The players of our market GNEP are producers, consumers, and the transport operator (TO). We start by discussing the producers (located at the nodes $u \in V$) that maximize their profit by solving the problem

$$\max_{s_u} \quad \pi_u s_u - c_{\mathrm{op},u}(s_u) \tag{3a}$$

$$\text{s.t.} \quad 0 \leq s_u \quad \text{for all } u \in V. \quad [\gamma_u^-] \tag{3b}$$

Here, π_u are exogenously given prices, which reflects the economic setting of perfect competition in which all players act as price takers. Dual variables are always denoted in brackets after the corresponding primal constraint.

The consumers (also located at nodes $u \in V$) maximize their benefit by solving

$$\max_{d_u} \quad \int_0^{d_u} p_u(x)\,\mathrm{d}x - \pi_u d_u. \tag{4a}$$

Here, we assume that demand is always positive at all nodes $u \in V$ without explicitly stating it as a constraint. This is an assumption that is often made in comparable equilibrium models—see, e.g., Gabriel et al. (2012)—and it is also not too restrictive in practice since non-demanding consumers can be excluded from the model.

Finally, we state the TO's model:

$$\max_{(q_a, K_a)_{a \in A}} \quad \sum_{a=(u,v) \in A} (\pi_v - \pi_u) q_a - \sum_{a \in A} c_{\mathrm{op},a}(q_a) - \sum_{a \in A} c_{\mathrm{inv},a}(K_a - K_a^-) \tag{5a}$$

$$\text{s.t.} \quad -K_a \leq q_a \leq K_a \quad \text{for all } a \in A, \quad [\beta_a^\pm] \tag{5b}$$

$$K_a^- \leq K_a \quad \text{for all } a \in A. \quad [\delta_a^-] \tag{5c}$$

The goal of the TO is to transport energy from low- to high-price areas and the TO thus maximizes the profit obtained by these price differences minus the transport and capacity investment costs. Constraint (5b) restricts the flow on arcs by arc capacities. Moreover, we impose lower bounds on the capacity in Constraint (5c).

Finally, all players consider their shared constraint

$$\sum_{a\in\delta^{\text{in}}(u)} q_a - \sum_{a\in\delta^{\text{out}}(u)} q_a - d_u + s_u = 0 \quad \text{for all } u \in V, \tag{6}$$

which models market clearing, i.e., energy flow balance at every node in the network.

This clearing constraint leads to the fact that variables of other players are part of every player's constraint set. Thus, the described setting is not a game but a generalized game in which we search for generalized Nash equilibria. For these GNEPs it is well-known that multiple equilibria exist (see, e.g., the survey by Facchinei and Kanzow 2007 or Facchinei et al. 2007), which means that we also obtain non-unique nodal prices that clear the market—an undesired situation. This is why one typically chooses a specific GNEP solution that is also a so-called variational equilibrium; see Harker (1991). Note that the shared constraints lead to the situation in which we have a single primal constraint at every node for which every player obtains a separate dual variable. In our convex setting, the variational equilibrium is then obtained by setting all these dual variables for the same primal constraint to the same value; see Harker (1991). Moreover, this specific variational GNEP solution can be obtained by solving the MCP that contains all Karush–Kuhn–Tucker (KKT) conditions for all players problems without the shared constraint and by adding this shared constraint afterward. This MCP is given by dual and primal feasibility,

$$-c'_{\text{op},u}(s_u) + \alpha_u + \gamma_u^- = 0 \quad \text{for all } u \in V, \tag{7a}$$

$$p_u(d_u) - \alpha_u = 0 \quad \text{for all } u \in V, \tag{7b}$$

$$-c'_{\text{op},a}(q_a) + \alpha_v - \alpha_u + \beta_a^- - \beta_a^+ = 0 \quad \text{for all } a = (u, v) \in A, \tag{7c}$$

$$-c'_{\text{inv},a}(K_a - K_a^-) + \beta_a^+ + \beta_a^- + \delta_a^- = 0 \quad \text{for all } a \in A, \tag{7d}$$

$$s_u \geq 0 \quad \text{for all } u \in V, \tag{7e}$$

$$-K_a \leq q_a \leq K_a \quad \text{for all } a \in A, \tag{7f}$$

$$K^{-}{}_{a} \leq K_a \quad \text{for all } a \in A, \tag{7g}$$

$$\sum_{a\in\delta^{\text{in}}(u)} q_a - \sum_{a\in\delta^{\text{out}}(u)} q_a - d_u + s_u = 0 \quad \text{for all } u \in V, \tag{7h}$$

and by the respective KKT complementarity constraints involving the non-negative dual variables for the corresponding inequality constraints.

Lastly, this MCP can be shown to be equivalent to the welfare maximization problem

$$\max_{d,s,K,q} \quad f(d, s, K, q) \tag{8a}$$

$$\text{s.t.} \quad \sum_{a\in\delta^{\text{in}}(u)} q_a - \sum_{a\in\delta^{\text{out}}(u)} q_a - d_u + s_u = 0 \quad \text{for all } u \in V, \qquad [\alpha_u] \tag{8b}$$

$$-K_a \leq q_a \leq K_a \quad \text{for all } a \in A, \qquad [\beta_a^{\pm}] \tag{8c}$$

$$0 \leq s_u \quad \text{for all } u \in V, \qquad [\gamma_u^-] \tag{8d}$$

$$K_a^- \le K_a \quad \text{for all } a \in A, \qquad [\delta_a^-] \tag{8e}$$

where corresponding dual variables are again denoted in brackets and where the objective function is defined as

$$\begin{aligned} f(d, s, K, q) := \sum_{u \in V} \int_0^{d_u} p_u(x)\, \mathrm{d}x - \sum_{u \in V} c_{\mathrm{op},u}(s_u) \\ - \sum_{a \in A} c_{\mathrm{inv},a}(K_a - K_a^-) - \sum_{a \in A} c_{\mathrm{op},a}(q_a). \end{aligned}$$

The latter models total social welfare that is defined as the difference of gross consumer surplus (first sum) and total costs of suppliers (second sum) and the TO (last two sums). The proof is easily obtained by comparing the KKT conditions of (8) with the MCP (7) and by identifying the dual variables α_u of the nodal balance constraints (8b) with the exogenously given prices π_u of the GNEP.

Note that all optimization problems considered so far are concave maximization problems over polyhedral feasible sets and that, thus, all KKT conditions are both necessary and sufficient. Due to the polyhedral feasible sets, no additional constraint qualification is required.

In what follows, we study the variational GNEP solutions by analyzing the welfare maximization problem or the equivalent MCP. First, we characterize the local market, i.e., the nodal prices α_u at each node u in the network.

Lemma 1 *The local price at node $u \in V$ is given by*

$$\alpha_u = p_u(d_u) \le c'_{\mathrm{op},u}(s_u).$$

If node u has active production, i.e., $s_u > 0$, then equality holds and prices are strictly positive:

$$p_u(d_u) = c'_{\mathrm{op},u}(s_u) > 0 \quad \textit{for all } u \in V \textit{ with } s_u > 0, \tag{9}$$

i.e., prices at nodes equal marginal costs of supply.

Proof The relations between $p(d_u)$ and $c'_{\mathrm{op},u}(s_u)$ follow directly from Conditions (7b) and (7a). Our standard assumptions entail $c'_{\mathrm{op},u}(s_u) > 0$. □

Since $c'_{\mathrm{inv},a}(K_a - K_a^-) > 0$, we also have

$$\beta_a^- > 0, \ \beta_a^+ > 0, \ \text{or } \delta_a^- > 0 \quad \text{for all } a \in A,$$

which readily implies (by KKT complementarity)

$$K_a = \max\left\{K_a^-, |q_a|\right\} \quad \text{for all } a \in A.$$

Thus, we either do not need to expand capacity or one of the flow bounds is active. In other words, capacity investment is realized as small as possible and as large as necessary in order to maximize welfare.

Next, we prove that the flow direction on arc $a = (u, v) \in A$ can be determined by the price difference between node u and v and that the absolute value of the price difference equals the sum of marginal transportation and capacity investment costs of the connecting arc. This is a result that can also be found in the literature; see, e.g., Cremer and Laffont (2002) for the same result in the context of a market power analysis on a two-node gas network.

Theorem 1 *Let $a = (u, v) \in A$ with $|q_a| > K_a^-$. Then*

$$\operatorname{sgn}(p_v(d_v) - p_u(d_u)) = \operatorname{sgn}(q_a)$$

and

$$|p_v(d_v) - p_u(d_u)| = c'_{\mathrm{inv},a}(K_a - K_a^-) + |c'_{\mathrm{op},a}(q_a)|. \tag{10}$$

Proof The last two dual feasibility Conditions (7c) and (7d) yield

$$\begin{aligned} c'_{\mathrm{inv},a}(K_a - K_a^-) + c'_{\mathrm{op},a}(q_a) &= \alpha_v - \alpha_u + 2\beta_a^- \quad \text{for all } a = (u, v) \in A, \\ c'_{\mathrm{inv},a}(K_a - K_a^-) - c'_{\mathrm{op},a}(q_a) &= \alpha_u - \alpha_v + 2\beta_a^+ \quad \text{for all } a = (u, v) \in A. \end{aligned}$$

If $q_a > 0$, then $q_a = K_a$ and hence the first equation together with (7b) yields the claim. The case $q_a < 0$ follows analogously from the second equation. □

We remark that the shadow price $\beta_a^\pm$ of capacity K_a equals marginal costs of capacity; see Condition (7d).

3.2 Second-best pricing

Since it will often be the case that Model (8) leads to deficits for the TO, it is reasonable to study an extension of this model, in which we add the break-even constraint

$$\sum_{u \in V} p_u(d_u)d_u - \sum_{u \in V} c'_{\mathrm{op},u}(s_u)s_u - \sum_{a \in A} c_{\mathrm{inv},a}(K_a - K_a^-) - \sum_{a \in A} c_{\mathrm{op},a}(q_a) \geq 0, \tag{11}$$

modeling that the TO's expenses can be covered. The first term of the constraint models the price that the consumers pay. From this we subtract the revenue of the producers (second term). The difference between these two terms must be large enough to cover the cost of the TO, namely investments in arcs (third term) and the costs of operating the network (fourth term). Revenue constraints like (11) are often discussed in the literature; see, e.g., Gabriel et al. (2013) and Ruiz et al. (2012), where the authors discuss similar constraints that ensure the non-negativity of the profits of the producers in their model. Similar to the cited papers, the prices obtained due to Constraint (11) guarantee that TO is willing to “remain in the market” because (11) prevents that the TO incurs a loss.

Revenue constraints like (11) are typically complicated to be implemented in practice because it is not entirely clear who decides on the investment and operating costs for the network. From a theoretical point of view, these costs in (11) are known since we consider a perfectly competitive market with complete information. In this setting, every agent uses true costs at the market. From a practical point of view, however, the situation is more complicated because the exact costs are not exactly known. However, estimates for, e.g., the costs of network expansion are often available in practice. As an example, the costs for the planned network expansion of the German electricity network are given in German TSOs (2014, 2017). To illustrate the availability of cost data for the German electricity sector we refer to Appendix A of the recent paper Ambrosius et al. (2018), where many sources of cost data are given.

If Constraint (11) is associated with the scalar dual variable $\eta \geq 0$, the dual feasibility conditions read

$$
\begin{aligned}
(1+\eta) p_u(d_u) - \alpha_u + \eta p_u'(d_u) d_u &= 0 \quad \text{for all } u \in V, && \text{(12a)}\\
-(1+\eta) c_{\text{op},u}'(s_u) + \alpha_u + \gamma_u^- - \eta c_{\text{op},u}''(s_u) s_u &= 0 \quad \text{for all } u \in V, && \text{(12b)}\\
-(1+\eta) c_{\text{op},a}'(q_a) + \alpha_v - \alpha_u - \beta_a^+ + \beta_a^- &= 0 \quad \text{for all } a = (u, v) \in A, && \text{(12c)}\\
-(1+\eta) c_{\text{inv},a}'(K_a - K_a^-) + \beta_a^+ + \beta_a^- + \delta_a^- &= 0 \quad \text{for all } a \in A. && \text{(12d)}
\end{aligned}
$$

Note that Constraint (11) might be nonlinear. Thus, additional constraint qualifications need to hold so that the KKT conditions again can be used. We now consider the cases in which the break-even constraint is binding or not. If the constraint is not binding, KKT complementarity yields $\eta = 0$ and the first-order conditions (12) equal the dual conditions of the problem without break-even constraint in (7). The case in which the break-even constraint is binding is more interesting. Adding (12a) and (12b) yields

$$
p_u(d_u) - c_{\text{op},u}'(s_u) = \frac{\eta}{1+\eta} \left(\frac{c_{\text{op},u}'(s_u)}{E_u^s} - \frac{p_u(d_u)}{E_u^d} \right) - \frac{\gamma_u^-}{1+\eta} \quad \text{for all } u \in V, \quad (13)
$$

instead of $p_u(d_u) - c_{\text{op},u}'(s_u) = -\gamma_u^-$; see (9). Here,

$$
E_u^s := \frac{c_{\text{op},u}'(s_u)}{c_{\text{op},u}''(s_u) s_u}, \quad E_u^d := \frac{p_u(d_u)}{p_u'(d_u) d_u}
$$

are the corresponding elasticities of supply and demand. Note that E_u^s is well-defined for $s_u > 0$ because $c_{\text{op},u}$ is assumed to be strongly convex. The results collected above lead to the following theorem, which is again a direct generalization of the results for two-node networks in Cremer and Laffont (2002).

Theorem 2 *Let $a = (u, v) \in A$ be an arc with $|q_a| > K_a^-$. Then*

$$
|p_v(d_v) - p_u(d_u)| = c_{\text{inv},a}'(K_a - K_a^-) + |c_{\text{op},a}'(q_a)|
$$

$$+\ \operatorname{sgn}(q_a)\frac{\eta}{1+\eta}\left(\frac{p_u(d_u)}{E_u^d}-\frac{p_v(d_v)}{E_v^d}\right).$$

Proof Subtract (12a) for adjacent nodes u, v and replace the resulting difference $\alpha_v - \alpha_u$ (if $q_a > 0$) or $\alpha_u - \alpha_v$ (if $q_a < 0$) using (12c). Reordering then yields the result. □

Thus, second-best price differences differ from their first-best counterparts (10) by inverse elasticity terms, which is expected from the classical Ramsey pricing rule (Ramsey 1927). Here, the economic assumption that the TO is regulated, but needs to cover its costs, is key: Without regulation, the TO would act as a monopolist—but since we assume that the TO is regulated, the prices are raised only by the amount that is needed to recover the costs. As in the classical Ramsey pricing rule, the price difference depends on inverse elasticities. That is, the more elastic the demand, the less markup is charged. Finally, (12d) yields

$$c'_{\mathrm{inv},a}(K_a - K_a^-) = \frac{\beta_a^+ + \beta_a^- + \delta_a^-}{1+\eta}$$

instead of $c'_{\mathrm{inv},a}(K_a - K_a^-) = \beta_a^+ + \beta_a^- + \delta_a^-$; see Condition (7d).

To sum up, all observed properties of the pricing in this section have a local character, i.e., they only depend on a single node or on a single arc and its adjacent nodes.

4 Pricing with potential-based network flow models

In this section we extend the models of the previous section by node potentials and possibly nonlinear constraints that couple potentials at nodes with flows on arcs. The first—and very important—difference is that the equivalence between the variational solutions of the GNEP and welfare maximal solutions does not hold in general in this setting. The reason is that the TO's model now contains nonconvex constraints and, thus, the KKT conditions are not sufficient anymore. Despite this problem, we analyze a potential-based extension of Model (8) in this section in order to shed light on the differences of the pricing mechanisms that are outcome of the corresponding welfare maximization problems. A detailed discussion of the market implications of the physical nonlinearities and the resulting differences between welfare optimal solutions and market equilibria can be found, e.g., in Grimm et al. (2019a) for the case of gas networks.

The main differences between the models of Sect. 3 and their potential-based counterparts used here are discussed in Sect. 4.1, where we start with the potential-based first-best case. In Sect. 4.2 we then also discuss second-best pricing. The case without transportation costs on arcs is discussed later in Sect. 5.2.

4.1 First-best pricing

We now consider the problem

$$\max_{d,s,K,q,\ell} \quad f(d,s,K,q) \tag{14a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a - d_u + s_u = 0 \quad \text{for all } u \in V, \qquad [\alpha_u] \tag{14b}$$

$$\ell_v - \ell_u + W_a(K_a) f(q_a) = 0 \quad \text{for all } a = (u,v) \in A, \qquad [\varepsilon_a] \tag{14c}$$

$$\ell_u^- \leq \ell_u \leq \ell_u^+ \quad \text{for all } u \in V, \qquad [\zeta_u^\pm] \tag{14d}$$

$$K_a^- \leq K_a \quad \text{for all } a \in A, \qquad [\delta_a^-] \tag{14e}$$

$$0 \leq s_u \quad \text{for all } u \in V, \qquad [\gamma_u^-] \tag{14f}$$

where $K_a^- > 0$ for all $a \in A$ in order to avoid a possible division by zero in $W_a(K_a)$; see Equation (2). Let us briefly explain this assumption using the example of gas networks, where capacity stands for pipe diameters; see Sect. 2. Constraint (14e) then means that we do not build completely new pipes but extend the capacity of already existing ones, where these existing pipes, obviously, need to have a non-zero diameter. Moreover, we assume that a constraint qualification like LICQ or MFCQ holds such that we can reasonably use KKT conditions.

In addition to linear flow models, we now have (possibly) nonlinear potential loss constraints (14c) that couple potentials on nodes with flows on arcs. Furthermore, we restrict node potentials by simple bounds (14d). This is common sense in practice since node potentials are often restricted due to technical issues. For instance, gas pressures at adjacent nodes u, v are typically restricted by the maximal pressure under which the connecting pipe $a = (u, v)$ can be operated. The potential loss constraints together with the bounds on the node potentials yield implicit bounds for the flows on arcs. This is the reason why we do not explicitly impose bounds on arc flows in this section. The main extension variable is the arc capacity K_a that determines the potential loss factor $W_a(K_a)$.

Before we discuss the first-order conditions of Problem (14) let us briefly discuss the meaning of the potential loss constraint (14c) in more detail. One way of interpreting these constraints is that they ensure that there cannot be any cycle flows in a potential-based network model. This is well-known for, e.g., lossless DC power flows in electricity networks or Weymouth-type pressure loss models in gas transport. On radial, i.e., tree-structured, transport networks, the potentials can be ignored and computed ex post for the model under consideration. For a more detailed discussion of this aspect see Krebs et al. (2018) for DC power flow networks and Grimm et al. (2019a) for gas transport models. As a consequence, computing flows on radial networks is easy (see, e.g., Lemma 2.3 in Robinius et al. 2019) whereas the computation of potential-based flows on meshed networks is much harder and typically requires to solve a system of nonlinear equations.

We now turn to the analysis of first-order conditions. Since the problems in this section are nonconvex, first-order conditions are only necessary. This has also been discussed in the light of the underlying GNEP at the very beginning of this section. Nevertheless, the KKT conditions hold at global optimal solutions. Dual feasibility conditions read

$$
\begin{aligned}
p_u(d_u) - \alpha_u &= 0 \quad \text{for all } u \in V, && (15a)\\
-c'_{\mathrm{op},u}(s_u) + \alpha_u + \gamma_u^- &= 0 \quad \text{for all } u \in V, && (15b)\\
-c'_{\mathrm{inv},a}(K_a - K_a^-) + \varepsilon_a W'_a(K_a) f(q_a) + \delta_a^- &= 0 \quad \text{for all } a \in A, && (15c)\\
-c'_{\mathrm{op},a}(q_a) + \alpha_v - \alpha_u + \varepsilon_a W_a(K_a) f'(q_a) &= 0 \quad \text{for all } a = (u, v) \in A, && (15d)\\
\sum_{a \in \delta^{\mathrm{in}}(u)} \varepsilon_a - \sum_{a \in \delta^{\mathrm{out}}(u)} \varepsilon_a + \zeta_u^- - \zeta_u^+ &= 0 \quad \text{for all } u \in V. && (15e)
\end{aligned}
$$

Here, we can already observe one of the major differences between potential-free and potential-based flow models. The dual feasibility conditions of the KKT conditions of the former models only contain equations that depend on a single node $u \in V$ or on a single arc $a = (u, v) \in A$. In the setting considered in this section, the dual condition (15e) connects node quantities with all arcs connected to this node. This is exactly the point where the local character of the pricing results for the potential-free model gets lost. For gas networks, this global behavior has also been reported in Midthun et al. (2009). Note further that these dual conditions corresponding to primal potentials again yield a dual flow problem with right-hand sides $\zeta_u^+ - \zeta_u^-$. In case of non-binding potential bounds, i.e., the case in which the newly introduced nonlinearities do not further restrict feasibility, this dual flow problem is a circulation.

In addition, the first four KKT conditions (15a)–(15d) not only relate the demand and cost quantities $p_u(d_u)$, $c'_{\mathrm{op},u}(s_u)$, $c'_{\mathrm{inv},a}(K_a - K_a^-)$, and $c'_{\mathrm{op},a}(q_a)$ to dual variables as it is was the case in (7) but also depend on the primal flows q_a. We discuss the implication of this fact later on in this section.

As in the linear case, the first two Conditions (15a) and (15b) imply the price equilibrium

$$p_u(d_u) = c'_{\mathrm{op},u}(s_u) \quad \text{for all } u \in V \text{ with } s_u > 0 \tag{16}$$

as well as $\alpha_u > 0$ for all $u \in V$ with $s_u > 0$. Thus, the local relation between price and marginal cost of production is the same as in the potential-free case; see Lemma 1. Moreover, from (15d) we know that

$$p_v(d_v) - p_u(d_u) = c'_{\mathrm{op},a}(q_a) - \varepsilon_a W_a(K_a) f'(q_a) \quad \text{for all } a = (u, v) \in A. \tag{17}$$

Theorem 3 *Let $a = (u, v) \in A$. If $K_a^- < K_a$ and $q_a \neq 0$, then*

$$\varepsilon_a = \frac{1}{W'_a(K_a) f(q_a)} c'_{\mathrm{inv},a}(K_a - K_a^-)$$

and

$$p_v(d_v) - p_u(d_u) = c'_{\mathrm{op},a}(q_a) - \frac{W_a(K_a) f'(q_a)}{W'_a(K_a) f(q_a)} c'_{\mathrm{inv},a}(K_a - K_a^-).$$

Note that the assumption $q_a \neq 0$ implies $f(q_a) \neq 0$ so that division by $W'_a(K_a) f(q_a)$ is well-defined. From the latter theorem we can deduce that if the lower capacity bound is not binding and the flow does not vanish, the direction of energy flow is determined by the sign of the price difference.

Note further that the pricing rule for potential-based flows given in Theorem 3 has, in contrast to the pricing rule (10) for potential-free models, an additional scaling of the capacity investment costs. This scaling is given by the nonlinear factor

$$\frac{W_a(K_a) f'(q_a)}{W_a'(K_a) f(q_a)},$$

which depends on the primal flow q_a and which can thus be interpreted as the worthiness of capacity expansion relative to the given flow situation. The price differences of Theorem 3 have the property that they are sufficient to induce the investment in capacity; see last term of the right-hand side. In the potential-free models, this term is independent of a specific flow situation, whereas the potential-based situation explicitly scales this investment term in dependence of the flow. This means, that an investment decision upon potential-based pricing better should use multiple-scenario setups in order to avoid curious investments that rely on a specific, but maybe unusual, flow situation.

As the previous discussion shows, the dependence of the pricing rule of the last theorem for the potential-based case more strongly depends on the flows in the network then the analogous rule (10) for the potential-free case. This makes cost allocation for using the transport network much more complicated. Consider, e.g., linear investment on transport costs for the potential-free case. In this setting, the price difference rule (10) is independent of the actual flow situation. For potential-based network flow models, this is not possible—except for the case of radial networks, where networks flows can be easily computed; see also our discussion of radial and meshed networks at the beginning of this section. For more details on cost allocation mechanisms in potential-based power networks we refer to Galiana et al. (2003) and Gil et al. (2005) and the references therein.

Corollary 1 *For every arc $a \in A$ at least one of the following statements holds:*

(1) $q_a = 0$,
(2) $K_a = K_a^-$,
(3) $\operatorname{sgn}(p_v(d_v) - p_u(d_u)) = -\operatorname{sgn}(\varepsilon_a) = \operatorname{sgn}(q_a)$.

Proof The claim directly follows from Condition (15c), Theorem 3, and the assumptions on W_a, f', $c'_{\mathrm{inv},a}$, and $c'_{\mathrm{op},a}$. □

The latter corollary states that, irrespective of the extension of the model towards nonlinear potential-based flow constraints, flow directions are still determined by price differences as it was the case for potential-free models; see Theorem 1, where the analogue conclusion is drawn. However, the more accurate physics model determines the feasibility of the flows, especially in cyclic networks, and prices thus need to be different compared the potential-free models.

The main effect observed in this section so far, which is different to the potential-free case, is the following: The primal economic data is not only connected with shadow prices, i.e., dual variables, but is also strongly related to primal flows. These primal flows are subject to potential gradients that are coupled throughout the entire network. Thus, pricing cannot be done locally in the network but changes in potentials or flows in some part of the network implicitly change the pricing everywhere else in the network as well. This effects will also be clearly visible in the case study of Sect. 6.

4.2 Second-best pricing

As for the potential-free models, we also discuss second-best pricing. To this end, we extend Model (14) by the corresponding break-even constraint (11), which is equipped with the scalar dual variable $\eta \geq 0$. Furthermore, we still assume that a sufficiently strong constraint qualification holds. The first-order conditions then read

$$
\begin{aligned}
(1+\eta) p_u(d_u) - \alpha_u + \eta p_u'(d_u) d_u &= 0 \quad \text{for all } u \in V, \\
-(1+\eta) c_{\mathrm{op},u}'(s_u) + \alpha_u + \gamma_u^- - \eta c_{\mathrm{op},u}''(s_u) s_u &= 0 \quad \text{for all } u \in V, \\
-(1+\eta) c_{\mathrm{inv},a}'(K_a - K_a^-) + \varepsilon_a W_a'(K_a) f(q_a) + \delta_a^- &= 0 \quad \text{for all } a \in A, \\
-(1+\eta) c_{\mathrm{op},a}'(q_a) + \alpha_v - \alpha_u + \varepsilon_a W_a(K_a) f'(q_a) &= 0 \quad \text{for all } a = (u,v) \in A, \\
\sum_{a=(v,u)\in A} \varepsilon_a - \sum_{a=(u,v)\in A} \varepsilon_a + \zeta_u^- - \zeta_u^+ &= 0 \quad \text{for all } u \in V.
\end{aligned}
$$

Obviously, we obtain Conditions (15) if the break-even constraint is not binding. Hence, we consider the binding case. In analogy to Sect. 3.2 we get

$$
p_u(d_u) - c_{\mathrm{op},u}'(s_u) = \frac{\eta}{1+\eta} \left(\frac{c_{\mathrm{op},u}'(s_u)}{E_u^s} - \frac{p_u(d_u)}{E_u^d} \right) - \frac{\gamma_u^-}{1+\eta} \quad \text{for all } u \in V.
$$

Again with the same technique as in Sect. 3.2 we obtain the price difference

$$
p_v(d_v) - p_u(d_u) = c_{\mathrm{op},a}'(q_a) - \frac{\varepsilon_a}{1+\eta} W_a(K_a) f'(q_a) + \frac{\eta}{1+\eta} \left(\frac{p_u(d_u)}{E_u^d} - \frac{p_v(d_v)}{E_v^d} \right)
$$

instead of (17). Moreover, it can be easily seen that Corollary 1 still holds.

Comparing the results with the linear case, we observe similarities. The additional inverse elasticity terms equal the corresponding terms in the potential-free models and the contribution of the potentials is scaled by a factor of $1/(1+\eta)$. This underlines that the main difference between the linear and nonlinear case are the additional terms that are introduced by node potentials. These again scale the terms in dependence of the specific flow situation; see also the discussion after Theorem 3.

5 The case without transportation costs

In this section we address the specific case without transportation costs on arcs. This case frequently appears in practice in subnetworks of the considered general networks. For instance, in gas networks, transportation costs mainly translate to the operational costs of compressor stations whereas gas flow through pipelines itself is costless. Since pipes typically outnumber all other network elements in gas transport networks, there are large subnetworks without transportation costs. A concept that is often used in this context is the one of price zones; see, e.g., the recent papers Grimm et al. (2016, 2017, 2019b), Kleinert and Schmidt (2019) and Krebs et al. (2018), as well as the references therein. This concept refers to connected subnetworks where prices are equal, which is obviously only possible in the absence of transportation costs on arcs. In this section, we formally derive these price zones from first-order conditions. For the potential-free case, we give a proof of existence of price zones that are given by flow-saturated arcs of the network. Moreover, for the potential-based case, we show that the case of non-binding node potential bounds leads to a single price zone covering the whole network.

5.1 The potential-free case

The main idea of this section is to use a different model formulation than the one used in the rest of this article; namely a variant based on the well-known theorems of Gale and Hoffman that give necessary and sufficient conditions when a network admits a feasible flow; see Schrijver (2003, Chapter 11). To simplify notation, we set $\delta(U) = \{a = (u, v) : u \in U, v \notin U\} \cup \{a = (u, v) : u \notin U, v \in U\}$ for every subset $U \subseteq V$.

$$\max_{d,s,K} \quad g(d, s, K) \tag{18a}$$

$$\text{s.t.} \quad -\sum_{a\in\delta(U)} K_a \le \sum_{u\in U} s_u - \sum_{u\in U} d_u \le \sum_{a\in\delta(U)} K_a \quad \text{for all } U \subseteq V, \quad [\alpha_U^{\pm}] \tag{18b}$$

$$K_a^- \le K_a \quad \text{for all } a \in A, \quad [\delta_a^-] \tag{18c}$$

$$0 \le s_u \quad \text{for all } u \in V, \quad [\gamma_u^-] \tag{18d}$$

$$\sum_{u\in V} s_u - \sum_{u\in V} d_u = 0, \quad [\lambda] \tag{18e}$$

where the objective is defined as

$$g(d, s, K) := \sum_{u\in V} \int_0^{d_u} p_u(x)\,\mathrm{d}x - \sum_{u\in V} c_{\mathrm{op},u}(s_u) - \sum_{a\in A} c_{\mathrm{inv},a}(K_a - K_a^-),$$

i.e., g equals f except for the missing transportation cost terms. It is known that we can restrict ourselves to those sets U that induce connected subgraphs of G; see Schrijver (2003, Chapter 11) again. The dual equations in this case are

$$
\begin{aligned}
p_u(d_u) - \sum_{U: u \in U} \alpha_U^- + \sum_{U: u \in U} \alpha_U^+ - \lambda &= 0 \quad \text{for all } u \in V, \\
-c'_{\mathrm{op},u}(s_u) + \sum_{U: u \in U} \alpha_U^- - \sum_{U: u \in U} \alpha_U^+ + \gamma_u^- + \lambda &= 0 \quad \text{for all } u \in V, \\
-c'_{\mathrm{inv},a}(K_a - K_a^-) + \sum_{U: a \in \delta(U)} \alpha_U^- + \sum_{U: a \in \delta(U)} \alpha_U^+ + \delta_a^- &= 0 \quad \text{for all } A \in A.
\end{aligned}
$$

If we add the first two dual conditions, we directly obtain that for all nodes with $s_u > 0$ it holds that $p_u(d_u) = c'_{\mathrm{op},u}(s_u)$. Furthermore, nodal prices can now be interpreted as follows: Every node set U that induces a connected subgraph, which we will refer to as a zone, has a price that is nonzero if and only if one of the capacity bounds for the induced cut is binding. A price at a node is simply the sum of all the prices of all zones that the node is part of. Especially, we can see that in the generic case nodes with the same price belong to a connected component of the graph. This shows that in this setting price zones can be obtained naturally from the first-order conditions.

For this we use the following definition of price zone.

Definition 1 Given a solution z of Problem (18), we say that a partition $\mathcal{Z} = \{Z_i\}_{i=1}^I$ partitions the node set V into *price zones*, if for all $Z \in \mathcal{Z}$ holds that for all nodes $u \in Z$ the prices $p_u(d_u)$ are equal. We also write $\mathcal{Z}(z)$ to emphasize the dependence on solution z.

The main idea is now to recover the price zones from the active constraints of the alternative formulation. For this we define the following partition; see also Krebs and Schmidt (2018) where the same concept is also used in the case of transport costs.

Definition 2 Given a solution z of Problem (8), we say that the partition $\mathcal{Z} = \{Z_i\}_{i=1}^I$ of the node set V is the *flow-induced partition*, if it is obtained the following way: Each Z_i is a connected component of the graph $\tilde{G}(z) = (V, E \setminus E^{\mathrm{sat}})$, where

$$
A^{\mathrm{sat}} := \left\{ a \in A : \exists U \subseteq V.\, a \in \delta(U) \text{ and } \sum_{a \in \delta(U)} K_a = \left| \sum_{u \in U} s_u - \sum_{u \in U} d_u \right| \right\}.
$$

With these concepts we are able to state an equivalent version of Problem (18).

Theorem 4 *Let $z^* := (d^*, s^*, K^*)$ be an optimal solution of Problem (18) and $\mathcal{Z}$ the flow-induced partition. Then,*

$$
\begin{aligned}
g(z^*) = \max_{d,s} \quad & g(d, s, K^*) && \text{(19a)} \\
\text{s.t.} \quad & \sum_{u \in Z} d_u - \sum_{u \in Z} s_u = \hat{K}_Z \quad \textit{for all } Z \in \mathcal{Z}, && \text{(19b)} \\
& 0 \le s_u \quad \textit{for all } u \in V, && \text{(19c)}
\end{aligned}
$$

where $\hat{K}_Z = \sum_{a \in \delta(Z)} K_a^*$ is the total in- or outflow of zone Z. This implies that $\mathcal{Z}$ is a partition into price zones.

Proof Let $z' = (s', d', K^*)$ be an optimal solution of Problem (19). As z^* is feasible for this problem, it is clear that $g(z^*) \leq g(z')$.

Now, for every $U \subseteq V$ define $\tau_U \in [0, 1]$ as follows: Choose τ_U maximal such that $z^{\tau_U} := (1 - \tau_U)z^* + \tau_U z'$ is feasible for the Constraint (18b) corresponding to U.

We observe that $\tau_U > 0$ for all $U \subseteq V$: This is clear if Constraint (18b) is not satisfied with equality for z^*. In the case that the constraint is satisfied with equality for z^*, we see that U can be written as a union of zones. Hence, the constraint is satisfied with equality for z' as well. So, we can choose $\tau_U = 1$ in this case.

Now set $\tau := \min_{U \subseteq V} \tau_U$. Then $z^\tau := (1 - \tau)z^* + \tau z'$ is feasible for Problem (18). Assume now, that $g(z^*) < g(z')$, then, as $\tau > 0$ holds, $g(z^\tau) > g(z^*)$ which contradicts the optimality of z^* for Problem (18). □

5.2 The potential-based case

In analogy to the last section we now analyze the case without transportation costs for the potential-based model, i.e., we neglect the terms $\sum_{a \in A} c_{\mathrm{op},a}(q_a)$ in the objective function of (14).

Proposition 1 *If no transportation costs are present, then there exists a matrix R such that $\varepsilon = R\zeta$, where $\zeta := \zeta^+ - \zeta^-$.*

Not surprisingly, the proof of Proposition 1 is along the lines of the classical proof of the computations of currents and voltages from Kirchhoff's laws; see, e.g., Bollobás (1998, Section II.3, esp. Theorem 11).

Proof of Proposition 1 Let M be the node-arc-incidence-matrix of G. Written in matrix form, Condition (15e) then reads $M\varepsilon = \zeta$. We arbitrarily choose a node $r \in V$, remove the row corresponding to r and obtain $\tilde{M}\varepsilon = \tilde{\zeta}$. If G is connected, then $\tilde{M}$ has full row rank. Let B be the set of arcs of a spanning tree in the graph and N be its complement. It is known that the square matrix $\tilde{M}_B$, where we pick all columns with arcs in B, is invertible. If we split $\tilde{M}$ using B and N the system reads

$$\tilde{M}_B \varepsilon_B + \tilde{M}_N \varepsilon_N = \tilde{\zeta} \iff \varepsilon_B = -\tilde{M}_B^{-1} \tilde{M}_N \varepsilon_N + \tilde{M}_B^{-1} \tilde{\zeta}.$$

This means, we can compute ε_B if we know ε_N. We define

$$C := \begin{pmatrix} -\tilde{M}_B^{-1} \tilde{M}_N \\ I \end{pmatrix}$$

and see that

$$C\varepsilon_N = \varepsilon - \begin{pmatrix} \tilde{M}_B^{-1} \\ 0 \end{pmatrix} \tilde{\zeta} \tag{20}$$

holds. Next, we define $P := \operatorname{diag}(W_a(K_a) f'(q_a))$. Since all diagonal entries are positive, P is invertible. As C is the fundamental cycle matrix of G, Condition (15d) can also be rewritten in matrix form and we obtain

$$C^\top P\left(C\varepsilon_N + \begin{pmatrix}\tilde{M}_B^{-1}\\ 0\end{pmatrix}\tilde{\zeta}\right) = 0.$$

Rearranging the terms then finally yields

$$C^\top P C\varepsilon_N = -C^\top P\begin{pmatrix}\tilde{M}_B^{-1}\\ 0\end{pmatrix}\tilde{\zeta}.$$

We observe that the matrix $C^\top PC$ is a square matrix of full rank as P is invertible and C has full column rank. Hence we can solve for ε_N and obtain

$$\varepsilon_N = -\left(C^\top PC\right)^{-1} C^\top P\begin{pmatrix}\tilde{M}_B^{-1}\\ 0\end{pmatrix}\tilde{\zeta}.$$

Together with (20) we obtain a matrix $\tilde{R}$ such that $\varepsilon = \tilde{R}\tilde{\zeta}$. After adding a zero-column, R is constructed as required. □

From Proposition 1 we can directly draw the following conclusions for the case that no potential bounds are binding.

Theorem 5 *If no transportation costs are present and no potential bound is binding, i.e.,* $\ell_u^- < \ell < \ell_u^+$ *for all* $u \in V$*, then*

(1) $p_v(d_v) = p_u(d_u)$ *for all* $u, v \in V$*,*
(2) $K_a = K_a^-$ *for all* $a \in A$*.*

Proof If no potential bound is active, $\zeta^+ = \zeta^- = 0$ holds. From Proposition 1 it then follows $\varepsilon = 0$ as well. The first claim follows from setting $\varepsilon = 0$ in Condition (15d). Then the claim holds for neighboring nodes. As G is connected the claim holds for all pairs of nodes.

For the second claim we set $\varepsilon = 0$ in (15c) and obtain $c'_{\mathrm{inv},a}(K_a - K_a^-) = \delta_a^-$. As $c'_{\mathrm{inv},a}(K_a - K_a^-) > 0$ is true by assumption, it follows that $\delta_a^- > 0$ holds. Then $K_a = K_a^-$ follows from KKT complementarity. □

The situation of binding potential bounds is much more difficult to analyze. The matrix R is typically dense, i.e., almost all of its entries are nonzero. This means a violation of a potential bound is not a local phenomenon, because a binding potential bound has an impact of almost all other nodes and arcs of the network. Thus, we can no longer talk of "a congested arc" as it is the case for potential-free energy flow models. In contrast, congestion is a global phenomenon. A case study illustrating these effects is given in Sect. 6.

Moreover, a result analogous to Theorem 5 does not hold for the case of transportation costs, e.g., price equality is then replaced by the price difference given in Theorem 3.

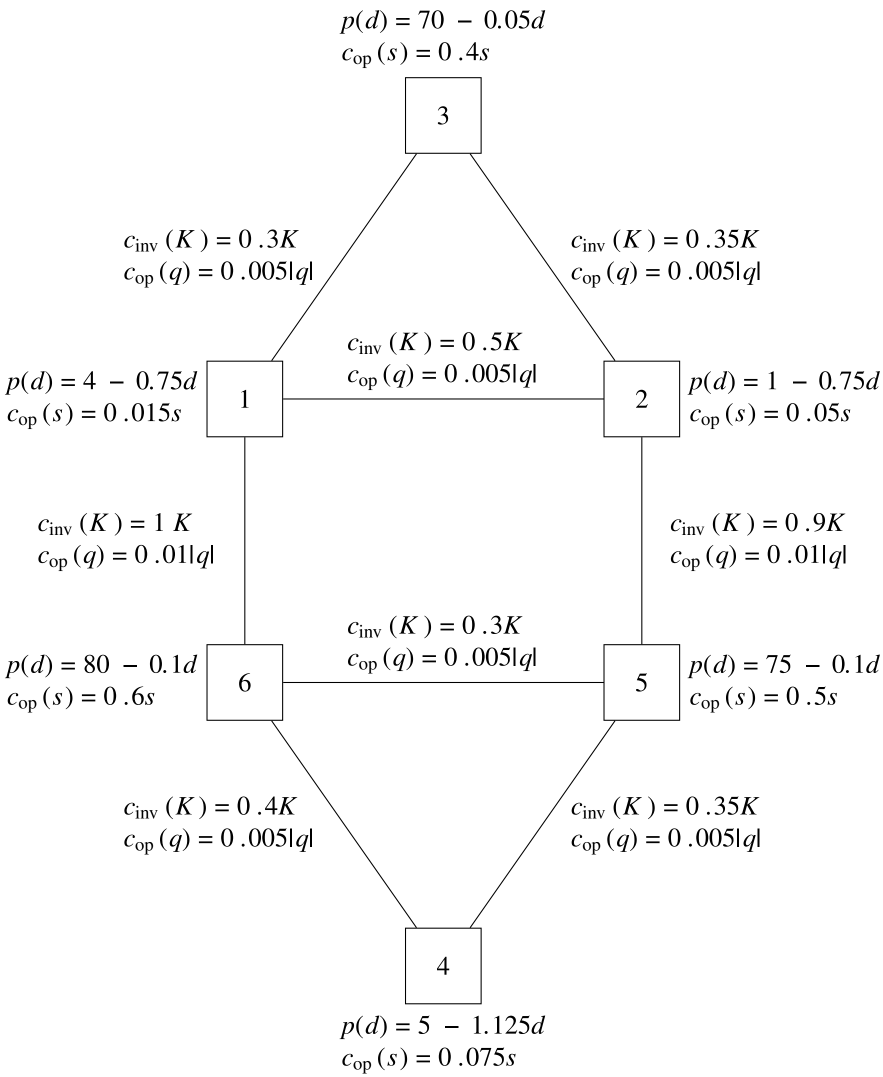

Fig. 1 Demand as well as investment, supply, and transportation costs for the 6-node network

6 Case study

In this section we compare the outcomes of the potential-free and potential-based first-best model on a cyclic six-node network. The topology of the network is taken from Chao and Peck (1998). We use the network topology given in the publication and modify the data of demand as well of supply, investment, and transportation costs as given in Fig. 1. Demand functions are modeled with affine-linear functions whereas investment, supply, and transportation costs are assumed to be (at least piecewise) linear. As already discussed in Sect. 4, the effects of potential loss constraints are only important for non-radial networks, which is why we choose the network in Fig. 1 that contains three fundamental cycles.

Before we start discussing the differences of the solutions to both models we mention that we are aware of the problem of calibrating the models so that they are comparable in a quantitative way. For instance, consider the meaning of "capacity" in both models: In the potential-free model, a unit of capacity expansion directly translates to one more unit of flow that can be sent through the corresponding arc. In contrast, capacities in the potential-based model correspond to, e.g., pipe diameters. Increased diameters also lead to increased amounts of flow that can be sent but there is a nonlinear relationship between capacity and flow instead of a linear one-to-one correspondence. Regarding the calibration of the model given in Fig. 1, we also want to note that it is not our purpose the construct a realistic instance but to set up an easy parameterization that allows to qualitatively compare the effects of potential-based physics models on pricing.

For the potential-free case, we solve Model (8) with lower capacity bounds $K_a^- = 0.1$ for all arcs $a \in A$. The potential-based first-best model (14) is instantiated in order to model a gas transport network. Thus, W_a is defined as in Equation (2), $f(q_a) = |q_a|q_a$, and capacities correspond to pipe diameters (again with lower bound 0.1).[1] Note that this model is nonlinear and nonconvex.

Both models are implemented in GAMS v24.3.3 (see McCarl 2009) and solved with BARON v14.0.3; see Tawarmalani and Sahinidis (2005). In all of our computations, BARON computes global optimal solutions. The GAMS source code of both models are publicly available and can be downloaded.[2]

Figure 2 shows the results of the linear model. The main demand nodes (downward triangles) are 3, 5, and 6 and the main suppliers (upward triangles) are nodes 1 and 4. Note that the main demand nodes do not supply whereas the main suppliers also have a small demand. Node 1 provides gas for node 3, node 2 is approximately autarkic and node 4 supplies node 5 and 6. Prices at node 1, 2, and 4 equal marginal costs of local supply since local supply is nonzero; see (9). Thus, differences between prices and marginal costs only exist at nodes without local supply, i.e., at nodes that are provided from other suppliers. These differences are determined by the dual variables γ^- of the lower supply bounds; see Lemma 1. Transport and investment costs are only caused by node pairs with flow that is induced by incentives due to demand and supply. Moreover, flow follows price differences; see Theorem 1. It can be also seen that larger price differences are caused by larger amounts of flow. More specifically, arcs that are not on a path from a main supply node to a main demand node do not cause any costs and, thus, costs can be distributed canonically between flow-inducing pairs of supply and demand nodes. For example, large transport and investment costs on the arc (1, 3) are caused by supply at node 1 that is used to serve the demand at node 3. The same applies to the arcs (4, 6) and (4, 5). All other arcs are only used at a minimum amount that is caused by the technical assumption of minimum capacities of 0.1.

[1] Potentials correspond to squared gas pressures and the potential bounds are chosen such that gas pressure is within 1 bar 70 bar. For the gas network model, all pipes have length $L = 25$ km and an inner roughness of $k = 0.05$ mm. Furthermore, the compressibility factor is set to 1, the constant mean temperature of gas is 283.15 K and the constant molar mass of gas is 18.05 kg kmol^{-1}.

[2] https://www.mso.math.fau.de/fileadmin/wima/data_members/schmidt/chao-peck.zip.

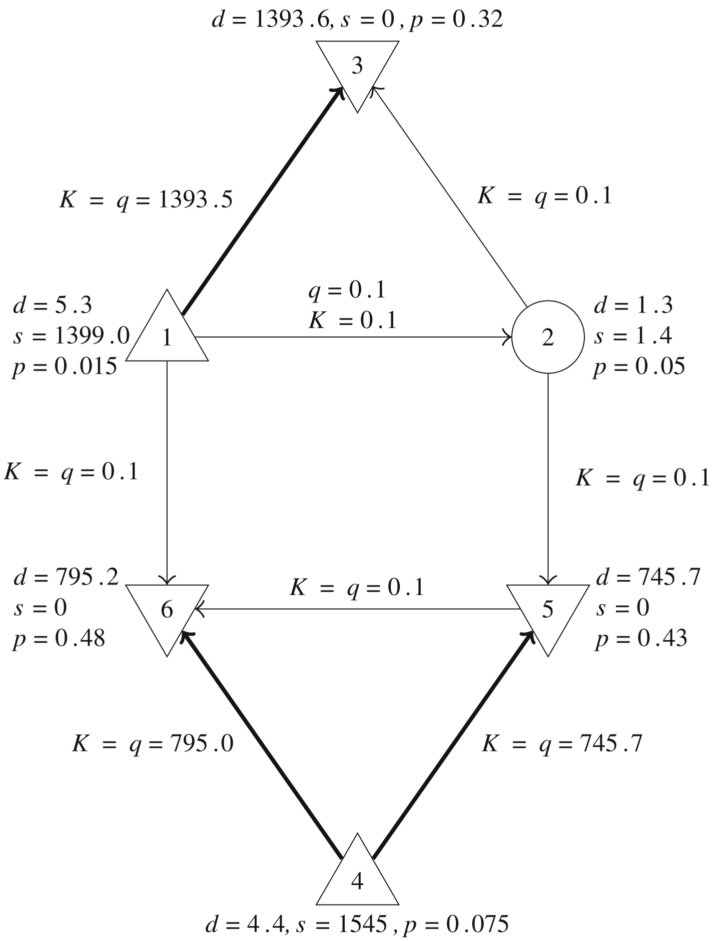

Fig. 2 Results for the potential-free model. Arcs with large flows are printed in bold. Flow directions are indicated by arrows, supply nodes by upward triangles, demand nodes by downward triangles and (approximately) autarkic nodes by circles

In summary, pricing of network access per node can be realized arc-wise considerations, i.e., by rules that are local in the network. In summary, all theoretical results can be seen in this case study.

The outcome of the potential-based model can be seen in Fig. 3. First, it turns out that the demand situation is qualitatively the same as for the potential-free model. We again have the main demand nodes 3, 5, and 6 (with no local supply as in the linear case). However, the supply situation significantly changed. Node 4 does not supply anymore yielding that node 1 is the only supplier. Node 2 is again, almost autarkic. As a consequence, the flow situation—and thus transport costs, capacity investments, and prices—significantly differs. The flow situation of the linear model shows two disconnected flow graphs (induced by significant flow), which lead to cheapest investment since capacity on arcs (1, 6) and (2, 5) are more expensive than

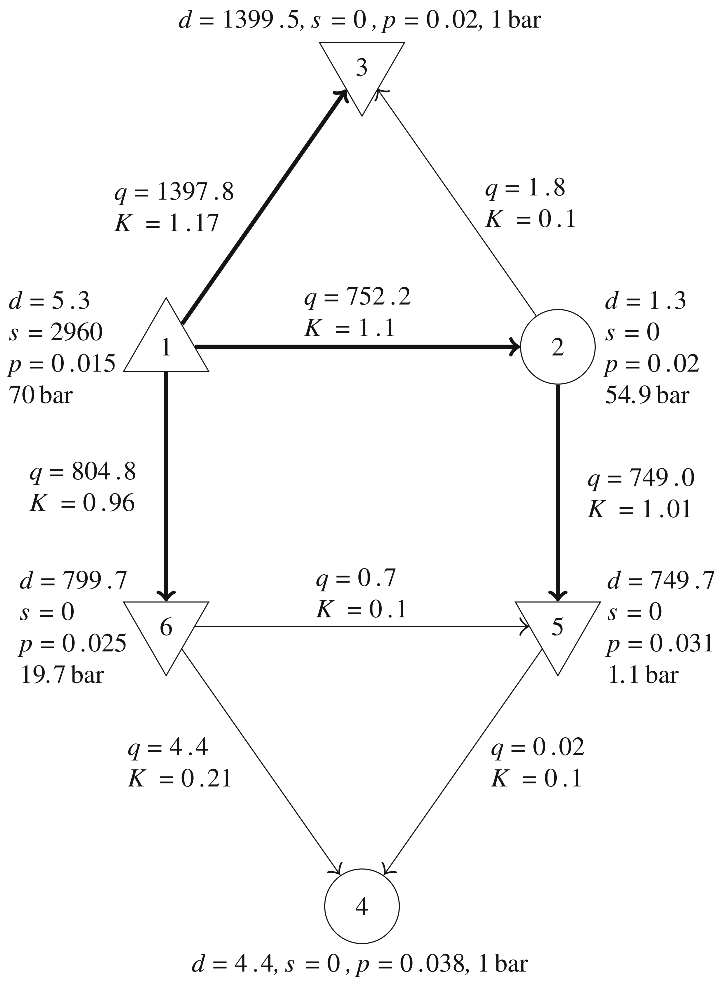

Fig. 3 Results for the potential-based model. Arcs with large flows are printed in bold. Flow directions are indicated by arrows, supply nodes by upward triangles, demand nodes by downward triangles and (approximately) autarkic nodes by circles

for all other arcs. However, these disconnected flows are not feasible for the potential-based flow model (14). This can also be seen in the solution to (14); see Fig. 3. The graph induced by gas flow now is connected, yielding a significant capacity expansion at both arcs $(1, 6)$ and $(2, 5)$. Note further that the cheaper path $(1, 6, 5)$ would not be feasible if one does not have the additional flow path $(1, 2, 5)$, too. This overall flow situation with enforced north-south flow allows that node 1, which is the cheapest supplier, provides all demand nodes with gas and the prices are always lower than in the solution of the linear and potential-free model. Node 1 is also the only node at which price equals marginal costs of local supply; see 16. We also see that the physical flow is in line with price gradients as stated in Corollary 1 and price differences are proportional to the amount of flow as stated in Theorem 3. Moreover, one sees that the potential-based flows lead to more homogeneously distributed prices throughout the network: The maximum price ratio, i.e., $\max_{u \in V} p_u / \min_{u \in V} p_u$, is 32 in the linear case whereas the corresponding value is 2.5 for the potential-based model. Thus, all nodes are affected by costs induced by only a few node pairs that would induce flow in the linear model due to price differences. For instance, prices at node 2 and 4 are affected although both nodes are almost autarkic, which is the computational realization of the effect that pricing does not have a local character for potential-based energy flow models; see also the discussions in Sect. 4.1.

Finally, this exemplary comparison of different energy flow models show that the outcomes of potential-free flow models significantly differ from their potential-based counterparts. The main difference in the discussed case study is that physical laws lead to line expansion that then allows the cheapest supplier to serve all demands in the network. In the potential-free case, the physical need for line expansion is not given, yielding a stronger regional supply.

7 Conclusion

In this paper, we presented a generalization of classical pricing frameworks for energy transport networks that uses potential-based physics models for the energy flow in the network. For the potential-free case, we obtained the result that the properties of the resulting pricing have a local character: Price differences are determined by the flow on the arc that connects these two nodes and the actions of the producers and consumers at the two nodes. This especially legitimates analyses of pricing mechanisms on potential-free networks that use very stylized networks like, e.g., two-node networks. In contrast, we also showed that this is not the case for potential-based networks because node potentials couple different regions of the network much stronger than it is the case for the standard nodal flow balance constraints that are also part of the potential-free models. This effect is particularly present in cyclic network structures, where feasible flows are mainly determined by the newly introduced constraints that couple node potentials with arc flows. Thus, significantly different pricing outcomes can be observed and we additionally provided a case study on a cyclic network that illustrates these differences.

There are many aspects of pricing in energy networks that we abstracted from. For example, we did not consider market power, which is a very important aspect of liberalized energy markets. We think that very interesting research questions can be obtained by combining market power considerations with potential-based energy flow models in networks, e.g., "Does potential-based physics models alleviate or enforce market power?" or "How does the consideration of potential-based models change the distribution of rents in imperfect markets?".

Finally, a deeper understanding of multiple-scenario settings is required for the case of potential-based flows since our theoretical results reveal that investment decisions based on a single flow situation may be undesirable.

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